Derivatives Pricing In An Incomplete Market
Part II - Minimize The Expected Squared Replication Error

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In Part I of this series we constructed the optimal hedge portfolio by minimizing the squared replication error. In Part II we will construct that hedge by minimizing the square of the **expected** replication error. In Part I we ignored the probabilities of each state of the world but in Part II we will not.

Our Hypothetical Problem From Part I

In Part I our goal was to build a hedge to hedge a short position in a put option. The table below presents the asset payoff matrix at time t given the state of the world at that time...

Table 1: Asset Payoff Matrix

SOTW	Description	Stock	Bond	Prob
1	Good economy	20.00	1.05	0.40
2	Average economy	10.00	1.05	0.50
3	Bad economy	5.00	1.05	0.10

The market prices of the stock and bond at time zero are \$12.00 and \$1.00, respectively.

Question: We have a put option on the stock with an exercise price of \$11.00 that can be exercised at time t. What is the value of the put option at time zero?

Constructing the Hedge Portfolio

In Part I we defined matrix $\mathbf{A}$ to be the payoff matrix applicable to our basis assets at time t. Using the data from our hypothetical problem above the equation for the payoff matrix is...

$$\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} 20.00 & 1.05 \\ 10.00 & 1.05 \\ 5.00 & 1.05 \end{bmatrix} \tag{1}$$

In Part I we defined vector $\vec{\mathbf{b}}$ to be the vector of payoffs on our focus asset, which is our put option, at time t. The equation for the focus asset payoff vector given the three states of the world is...

$$\vec{\mathbf{b}} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 9.00 \\ 4.00 \\ 0.00 \end{bmatrix} \tag{2}$$

In Part I we defined vector $\vec{\mathbf{v}}$ to be the vector of asset prices at time zero. Using the data from our hypothetical problem above the equation for the focus asset payoff vector given the three states of the world is...

$$\vec{\mathbf{v}} = \begin{bmatrix} \text{Stock} \\ \text{Bond} \end{bmatrix} = \begin{bmatrix} 12.00 \\ 1.00 \end{bmatrix} \tag{3}$$

We will define vector $\vec{\mathbf{p}}$ to be the vector of state probabilities. Using the data from our hypothetical problem above the equation for this vector is...

$$\vec{\mathbf{p}} = \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0.40 \\ 0.50 \\ 0.10 \end{bmatrix} \tag{4}$$

In Part I we defined vector $\vec{\mathbf{w}}$ to be the vector of asset weights. This is the vector that we want to solve for. The equation for the asset weight vector is...

$$\vec{\mathbf{w}} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} \tag{5}$$

In Part I we defined the vector $\vec{\mathbf{e}}$ to be the vector of hedging (i.e. replication) errors. Using Equations (1), (2) and (5) above the equation from Part I that defines our hedge is...

$$\mathbf{A}\vec{\mathbf{w}} = \vec{\mathbf{b}} + \vec{\mathbf{e}} \text{ ...where... } \vec{\mathbf{e}} = \mathbf{A}\vec{\mathbf{w}} - \vec{\mathbf{b}} \tag{6}$$

Minimizing the Expected Squared Replication Error

To minimize the expected squared replication error we need to construct a hedge portfolio at time zero that has a $w_1 \times$ stock price dollar position in the stock and a $w_2 \times$ bond price dollar position in the bond. To do this we must first solve for vector $\vec{\mathbf{w}}$, which is the vector of asset weights. We will start by multiplying both sides of Equation (18) above by the inverse of the product of the transpose of matrix A and matrix A. This statement in equation form is...

$$\vec{\mathbf{e}} = \begin{bmatrix} a_{11}w_1 + a_{12}w_2 - b_1 \\ a_{21}w_1 + a_{22}w_2 - b_2 \\ a_{31}w_1 + a_{32}w_2 - b_3 \end{bmatrix} \tag{7}$$

In Part II we want to minimize the expected squared replication error. Using Equation (4) above we will modify Equation (6) above such that the error vector becomes...

$$\vec{\mathbf{e}}^* = \begin{bmatrix} \sqrt{p_1} \times (a_{11}w_1 + a_{12}w_2 - b_1) \\ \sqrt{p_2} \times (a_{21}w_1 + a_{22}w_2 - b_2) \\ \sqrt{p_3} \times (a_{31}w_1 + a_{32}w_2 - b_3) \end{bmatrix} \tag{8}$$

To get the error vector as defined by Equation (8) above we will modify matrix $\mathbf{A}$ as defined by Equation (1) above as follows...

$$\mathbf{A}^* = \begin{bmatrix} \sqrt{p_1} \times a_{11} & \sqrt{p_1} \times a_{12} \\ \sqrt{p_2} \times a_{21} & \sqrt{p_2} \times a_{22} \\ \sqrt{p_3} \times a_{31} & \sqrt{p_3} \times a_{32} \end{bmatrix} = \begin{bmatrix} \sqrt{0.40} \times 20.00 & \sqrt{0.40} \times 1.05 \\ \sqrt{0.50} \times 10.00 & \sqrt{0.50} \times 1.05 \\ \sqrt{0.10} \times 5.00 & \sqrt{0.10} \times 1.05 \end{bmatrix} \tag{9}$$

To get the error vector as defined by Equation (8) above we will modify vector $\vec{\mathbf{b}}$ as defined by Equation (2) above as follows...

$$\vec{\mathbf{b}}^* = \begin{bmatrix} \sqrt{p_1} \times b_1 \\ \sqrt{p_2} \times b_2 \\ \sqrt{p_3} \times b_3 \end{bmatrix} = \begin{bmatrix} \sqrt{0.40} \times 9.00 \\ \sqrt{0.50} \times 4.00 \\ \sqrt{0.10} \times 0.00 \end{bmatrix} \tag{10}$$

Using Equations (8), (9) and (10) above we will rewrite the vector of hedging errors as defined by Equation (6) above as...

$$\mathbf{A}^*\vec{\mathbf{w}} = \vec{\mathbf{b}}^* + \vec{\mathbf{e}}^* \text{ ...where... } \vec{\mathbf{e}}^* = \mathbf{A}^*\vec{\mathbf{w}} - \vec{\mathbf{b}}^* \tag{11}$$

We will define the variable $ESRE$ to be the expected squared replication error and the variable m to be the number of states of the world. Using Equation (8) above the equation for the expected squared replication error is...

$$SRE = \sum_{i=1}^{m} e_i^2 = p_1\,(a_{11}w_1 + a_{12}w_2 - b_1)^2 + p_2\,(a_{21}w_1 + a_{22}w_2 - b_2)^2 + p_3\,(a_{31}w_1 + a_{32}w_2 - b_3)^2 \tag{12}$$

The derivative of Equation (12) above with respect to w_1 is...

$$\frac{\delta\,SRE}{\delta\,w_1} = 2\,p_1\,a_{11}\,(a_{11}w_1 + a_{12}w_2 - b_1) + 2\,p_2\,a_{21}\,(a_{21}w_1 + a_{22}w_2 - b_2) + 2\,p_3\,a_{31}\,(a_{31}w_1 + a_{32}w_2 - b_3) \tag{13}$$

The derivative of Equation (12) above with respect to w_2 is...

$$\frac{\delta\,SRE}{\delta\,w_2} = 2\,p_1\,a_{12}\,(a_{11}w_1 + a_{12}w_2 - b_1) + 2\,p_2\,a_{22}\,(a_{21}w_1 + a_{22}w_2 - b_2) + 2\,p_3\,a_{32}\,(a_{31}w_1 + a_{32}w_2 - b_3) \tag{14}$$

To minimize the sum of expected squared replication errors we set the derivatives of Equations (13) and (14) above to zero and solve those equations simultaneously. Using Appendix Equations (22) and (23) below the equations for the derivatives of the sum of expected squared replication errors set to zero are...

$$\begin{aligned} 0 &= (p_1 a_{11} a_{11} + p_2 a_{21} a_{21} + p_3 a_{31} a_{31}) w_1 + (p_1 a_{11} a_{12} + p_2 a_{21} a_{22} + p_3 a_{31} a_{32}) w_2 - (p_1 a_{11} b_1 + p_2 a_{21} b_2 + p_3 a_{31} b_3) \\ 0 &= (p_1 a_{12} a_{11} + p_2 a_{22} a_{21} + p_3 a_{32} a_{31}) w_1 + (p_1 a_{12} a_{12} + p_2 a_{22} a_{22} + p_3 a_{32} a_{32}) w_2 - (p_1 a_{12} b_1 + p_2 a_{22} b_2 + p_3 a_{32} b_3) \end{aligned} \tag{15}$$

Using Equations (13) and (14) above the second derivatives of Equation (12) above with respect to w_1 and w_2 are...

$$\frac{\delta^2\, SRE}{\delta\, w_1^2} = 2\Bigg(p_1\, a_{11}^2 + p_2\, a_{21}^2 + p_3\, a_{31}^2\Bigg) > 0 \text{ ...and... } \frac{\delta^2\, SRE}{\delta\, w_2^2} = 2\Bigg(p_1\, a_{12}^2 + p_2\, a_{22}^2 + p_3\, a_{32}^2\Bigg) > 0 \tag{16}$$

Since the second derivatives are greater than zero we know that at the point where the first derivatives are zero marks the minimum point (i.e. SRE decreases to its minimum point and then increases thereafter) rather than the maximum point.

Note that we can rewrite the system of equations in Equation (15) as...

$$\begin{aligned} (p_1 a_{11} a_{11} + p_2 a_{21} a_{21} + p_3 a_{31} a_{31}) w_1 + (p_1 a_{11} a_{12} + p_2 a_{21} a_{22} + p_3 a_{31} a_{32}) w_2 &= p_1 a_{11} b_1 + p_2 a_{21} b_2 + p_3 a_{31} b_3 \\ (p_1 a_{12} a_{11} + p_2 a_{22} a_{21} + p_3 a_{32} a_{31}) w_1 + (p_1 a_{12} a_{12} + p_2 a_{22} a_{22} + p_3 a_{32} a_{32}) w_2 &= p_1 a_{11} b_1 + p_2 a_{21} b_2 + p_3 a_{31} b_3 \end{aligned} \tag{17}$$

Note that we can express the system of equations in Equation (17) above in matrix:vector notation as...

$$\mathbf{A}^{*T}\mathbf{A}^{*}\,\vec{\mathbf{w}} = \mathbf{A}^{*T}\,\vec{\mathbf{b}}^{*} \tag{18}$$

The Solution to the Hypothetical Problem

As was the case in Part I the equation that minimizes the expected squared replication error is...

$$\Bigg(\mathbf{A}^{*T}\mathbf{A}^{*}\Bigg)^{-1}\mathbf{A}^{*T}\mathbf{A}^{*}\,\vec{\mathbf{w}} = \Bigg(\mathbf{A}^{*T}\mathbf{A}^{*}\Bigg)^{-1}\mathbf{A}^{*T}\,\vec{\mathbf{b}}^{*} \text{ ...such that... } \vec{\mathbf{w}} = \Bigg(\mathbf{A}^{*T}\mathbf{A}^{*}\Bigg)^{-1}\mathbf{A}^{*T}\,\vec{\mathbf{b}}^{*} \tag{19}$$

Using Equations (9) and (10) above the solution to Equation (19) above, which is the vector of asset weights, is...

$$\vec{\mathbf{w}} = \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} -0.23 \\ 3.96 \end{bmatrix} \tag{20}$$

Using Equations (3) and (20) above the equation for put option price at time zero is...

$$\text{Put option price} = \vec{\mathbf{w}}^T\vec{\mathbf{v}} = \begin{bmatrix} -0.23 \\ 3.96 \end{bmatrix}^T \begin{bmatrix} 12.00 \\ 1.00 \end{bmatrix} = 1.24 \tag{21}$$

Appendix

A. To minimize the squared replication error (SRE) we first set Equation (13) above, which is the first derivative of the sum of squared errors with respect to w_1, equal to zero....

$$\begin{aligned} 2\,p_1 a_{11}\,(a_{11} w_1 + a_{12} w_2 - b_1) + 2\,p_2 a_{21}\,(a_{21} w_1 + a_{22} w_2 - b_2) + 2\,p_3 a_{31}\,(a_{31} w_1 + a_{32} w_2 - b_3) &= 0 \\ p_1 a_{11}\,(a_{11} w_1 + a_{12} w_2 - b_1) + p_2 a_{21}\,(a_{21} w_1 + a_{22} w_2 - b_2) + p_3 a_{31}\,(a_{31} w_1 + a_{32} w_2 - b_3) &= 0 \\ (p_1 a_{11} a_{11} + p_2 a_{21} a_{21} + p_3 a_{31} a_{31}) w_1 + (p_1 a_{11} a_{12} + p_2 a_{21} a_{22} + p_3 a_{31} a_{32}) w_2 - (p_1 a_{11} b_1 + p_2 a_{21} b_2 + p_3 a_{31} b_3) &= 0 \end{aligned} \tag{22}$$

B. To minimize the squared replication error (SRE) we next set Equation (14) above, which is the first derivative of the sum of squared errors with respect to w_2, equal to zero....

$$\begin{aligned} 2\,p_1 a_{12}\,(a_{11} w_1 + a_{12} w_2 - b_1) + 2\,p_2 a_{22}\,(a_{21} w_1 + a_{22} w_2 - b_2) + 2\,p_3 a_{32}\,(a_{31} w_1 + a_{32} w_2 - b_3) &= 0 \\ p_1 a_{12}\,(a_{11} w_1 + a_{12} w_2 - b_1) + p_2 a_{22}\,(a_{21} w_1 + a_{22} w_2 - b_2) + p_3 a_{32}\,(a_{31} w_1 + a_{32} w_2 - b_3) &= 0 \\ (p_1 a_{12} a_{11} + p_2 a_{22} a_{21} + p_3 a_{32} a_{31}) w_1 + (p_1 a_{12} a_{12} + p_2 a_{22} a_{22} + p_3 a_{32} a_{32}) w_2 - (p_1 a_{12} b_1 + p_2 a_{22} b_2 + p_3 a_{32} b_3) &= 0 \end{aligned} \tag{23}$$